On the Geometry of Line Elements in the Plane with Reference to Osculating Circles.

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Introduction.

The position of a line element in the plane is fixed by three coördinates (x, y, p). If we regard these coördinates as functions of a single parameter t,

$$x = x(t), \quad y = y(t), \quad p = p(t),$$

we have an analytic definition of a *line-element locus of* ∞^1 *elements.* We associate with this line-element locus a *point locus* defined by

$$x = x(t), \quad y = y(t)$$

and a *line locus,* which is the envelope of the one-parameter family of lines defined by

$$y - y(t) = p(t)\,[x - x(t)].$$

If the point locus and the line locus of a line-element locus coincide, we call the latter a *union,* the analytic condition for such being

$$dy - p\,dx = 0. \tag{1}$$

There are two particular unions of especial interest — the *point union* and the *line union* — for the singularities of a general union arise from its extraordinary relation to these two.*

Now, if we regard the coördinates of a line element as functions of two independent parameters

$$x = x(t, s), \quad y = y(t, s), \quad p = p(t, s), \tag{2}$$

we have a *line-element locus of* ∞^2 *elements.* Such a locus may always be resolved

* Two unions are tangent when they have a line element in common, and osculate when they have two consecutive line elements in common. If a point union or a line union has more than one line element in common with a general union, that point or line is *singular* with respect to the general union; thus, for example, a double point or a double tangent; or if the two line elements are consecutive, a cusp or an inflectional tangent. That is, a cusp is a point which *osculates* a curve.

in infinitely many ways into ∞^1 loci of ∞^1 elements, but in one way only into ∞^1 unions. If we eliminate the two parameters from equations (2), we obtain in general* an *ordinary differential equation of the first order*,

$$f(x, y, p) = 0,$$

whose integral curves are the ∞^1 unions referred to. We associate then in general an ordinary differential equation of the first order with a line-element locus of ∞^2 elements.

The object of this paper is to obtain theorems on the two types of line-element loci just considered. The problem in the plane is changed to one in space by a transformation due to Lie,† viz.,

$$T \begin{cases} x = i\left(Z + \dfrac{X}{Y}\right), \\ y = Z - \dfrac{X}{Y}, \\ p = \dfrac{i(1 - Y^2)}{1 + Y^2}, \end{cases}$$

which sets up a correspondence between the points (X, Y, Z) of space and the line elements (x, y, p) in the plane. Hence, curves and surfaces give rise to line-element loci of ∞^1 and of ∞^2 elements respectively.

Furthermore, under the transformation T the equation (1) becomes

$$X dY - Y dX + dZ = 0, \tag{3}$$

and hence only those curves in space which are integral curves of this Pfaffian equation give rise to unions in the plane. But these curves are those of a linear line complex whose axis is the axis of Z; that is, the tangents of all such curves constitute an assemblage of ∞^3 lines whose Plückerian line coördinates satisfy a linear relation. These coördinates are defined by the two-rowed determinants of the matrix

$$\begin{Vmatrix} X & Y & Z & 1 \\ dX & dY & dZ & 0 \end{Vmatrix},$$

viz.,

$$\left.\begin{aligned} p_1 &= Z dY - Y dZ, & p_4 &= dX, \\ p_2 &= X dZ - Z dX, & p_5 &= dY, \\ p_3 &= Y dX - X dY, & p_6 &= dZ, \end{aligned}\right\} \tag{4}$$

* Unless the ∞^1 unions are all point unions.

† "Geometrie der Berührungstransformationen," p. 247. The transformation given above differs slightly from Lie's transformation by the projective transformation

$$\bar{X} = 4X; \quad \bar{Y} = \frac{Y}{2}; \quad \bar{Z} = 2Z,$$

the change being made to facilitate the analysis.

which satisfy the identical relation

$$p_1 p_4 + p_2 p_5 + p_3 p_6 \equiv 0. \tag{5}$$

The linear equation which the line coördinates satisfy is with reference to (3) and (4) easily seen to be

$$p_3 = p_6 .$$

We shall hereafter refer to this as the fundamental complex Γ. The *aequatio directrix* of the equivalent null-system is

$$Y_0 X - X_0 Y - Z + Z_0 = 0. \tag{7}$$

The geometry in space which we shall interpret in the plane by means of the transformation T is the projective geometry within the fundamental complex Γ; that is, the geometry whose group is the G_{10} of all linear transformations under which the ∞^3 lines of Γ form an invariant system, or in other words, all linear transformations which leave the equation (3) invariant.

It will be shown later* that under T the lines of Γ transform into *circular unions* in the plane, and we have already stated that equation (3) transforms into (1). Hence, the geometry in the plane is that geometry whose group is the Γ_{10} of all contact transformations which leave the system of ∞^3 circles in the plane invariant. Knowing then the distinct types of curves and surfaces under G_{10} in space, we are in a position to classify line-element loci of ∞^1 and of ∞^2 elements with respect to Γ_{10} in the plane.

§ 1. *Line-Element Loci of ∞^1 Elements.*

Consider a general skew curve in space defined by

$$X = f(t), \qquad Y = g(t), \qquad Z = h(t).$$

Then, under the transformation T,

$$x = i\left(Z + \frac{X}{Y}\right),$$

$$y = Z - \frac{X}{Y},$$

$$p = \frac{i(1 - Y^2)}{1 + Y^2},$$

* See also "Geometrie der Berührungstransformationen."

the corresponding line-element locus in the plane is

$$x = i\left(h + \frac{f}{g}\right),$$
$$y = h - \frac{f}{g},$$
$$p = \frac{i\,(1 - g^2)}{1 + g^2}.$$

The point locus is defined parametrically by the first two of these equations, and the line locus is defined by the envelope of the family of lines

$$(1 + g^2)\,y - i\,(1 - g^2)\,x = 2\,(h - f g),$$

which envelope has the following parametric equations:

$$x = i\left(h + \frac{f}{g}\right) - \frac{i\,(1 + g^2)}{2\,gg'}(f g' - g f' + h'),$$
$$y = h - \frac{f}{g} + \frac{1 - g^2}{2\,gg'}(f g' - g f' + h').$$

The equations defining the point locus and the line locus are identical when and only when

$$f g' - g f' + h \equiv 0,$$

which is the condition that the original skew curve be a curve of the complex Γ, or an integral curve of equation (3). This agrees with our statement that only those curves of Γ give rise to unions in the plane. We shall refer to such curves hereafter simply as *complex* curves.

THEOREM 1: *The complex curves of space and such curves only give rise to unions in the plane.*

Consider now a general line in space. Its six coördinates enter into the equations of its projecting planes as follows:

$$\left.\begin{aligned} p_6 X &= p_4 Z + p_2, \\ p_6 Y &= p_5 Z - p_1, \\ p_5 X &= p_4 Y - p_3. \end{aligned}\right\} \tag{8}$$

The equations of the line may then be written in the following parametric form:

$$\left.\begin{aligned} X &= \frac{p_4 t - p_3}{p_5}, \\ Y &= t, \\ Z &= \frac{p_6 t + p_1}{p_5}. \end{aligned}\right\} \tag{9}$$

The corresponding line-element locus is

$$x = \frac{i}{p_5 t}\,[p_6 t^2 + (p_1 + p_4)\,t - p_3],$$
$$y = \frac{1}{p_5 t}\,[p_6 t^2 + (p_1 - p_4)\,t + p_3],$$
$$p = i\left[\frac{1 - t^2}{1 + t^2}\right].$$

The point locus, found by eliminating the parameter from the first two equations, has the equation

$$p_5^2\,(x^2 + y^2) - 2\,i p_5\,(p_1 + p_4)\,x - 2\,p_5\,(p_1 - p_4)\,y + 4\,p_2 p_5 = 0.^* \qquad (10)$$

The line locus is the envelope of the family of lines

$$[p_5\,(y + i\,x) + 2p_4]\,t^2 - 2\,(p_3 + p_6)\,t + [p_5\,(y - i\,x) - 2\,p_1] = 0.$$

Eliminating t from this equation and its derivative, we obtain

$$p_5^2(x^2 + y^2) - 2\,i p_5\,(p_1 + p_4)\,x - 2p_5\,(p_1 - p_4)\,y - [(p_3 + p_6)^2 + 4\,p_1 p_4] = 0. \quad (11)$$

Hence,

Theorem 2: *The point locus and the line locus of a general line in space are both circles having the same center.*

Since the coördinates of the reciprocal polar line under Γ of a given line are found by interchanging the p_3 and p_6 coördinates of the latter, we can state by reference to (10) and (11)

Theorem 3: *Reciprocal polar lines of* Γ *have the same point locus and the same line locus.*

Setting $p_3 = p_6$, the general line becomes a complex line and equations (10) and (11) both reduce† to

$$p_5\,(x^2 + y^2) - 2\,i\,(p_1 + p_4)\,x - 2\,(p_1 - p_4)\,y + 4\,p_2 = 0. \qquad (12)$$

Theorem 4: *There is a one-to-one correspondence between the complex lines in space and the circular unions in the plane.*

Consider those complex lines which are perpendicular to the axis of Z. They are defined by

$$\begin{aligned} Z &= k, \\ X &= KY. \end{aligned} \qquad (13)$$

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* The constant term has been simplified by means of (5).

† The constant term in (11) becomes $-(4\,p_6^2 + 4\,p_1 p_4) = 4\,p_2 p_5$, by (5).

Hence, the corresponding unions are characterized by

$$x = i\,(k + K) = \text{const.},$$
$$y = k - K = \text{const.},$$

evidently point unions. We shall refer to such lines as *P lines.*

THEOREM 5: *There is a one-to-one correspondence between the P lines in space and the point unions in the plane.*

Consider those complex lines which are perpendicular to the axis of Y. They are defined by

$$\begin{aligned} Y &= h, \\ Z - h\,X &= H. \end{aligned} \tag{14}$$

The corresponding unions are characterized by

$$p = \frac{i\,(1 - h^2)}{1 + h^2} = \text{const.},$$

evidently line unions. Hence, we shall refer to these lines as *L lines.*

THEOREM 6: *There is a one-to-one correspondence between the L lines in space and the line unions in the plane.*

We tabulate this correspondence as follows:

Space.	*Plane.*
Point.	Line element.
Complex curve.	Union.
Complex line.	Circular union.
P line.	Point union.
L line.	Line union.

Consider now two general curves in space. If these curves intersect, their corresponding line-element loci have a line element in common; hence

THEOREM 7: *If two general curves in space intersect, their corresponding point loci intersect and their corresponding line loci have a common tangent.*

Let the two general curves be tangent; then their corresponding line-element loci have two consecutive line elements in common; hence

THEOREM 8: *If two general curves in space are tangent, their corresponding line loci as well as their corresponding point loci are also tangent.*

Consider two complex curves. If they intersect, their corresponding unions have a line element in common; hence

THEOREM 9: *If two complex curves intersect, their corresponding unions are tangent.*

Apply this theorem to the interpretation of all the complex lines lying in a given plane. Since one of these lines is a P line and one an L line, and since all pass through a common point, we state

THEOREM 10: *Corresponding to all the complex lines lying in a given plane, we have a pencil of tangent circular unions, including the point union and the line union determined by the common line element.*

If two complex curves are tangent, their unions have two consecutive line elements in common; hence

THEOREM 11: *If two complex curves are tangent, their corresponding unions osculate;* and, in general,

THEOREM 12: *If two complex curves have contact of the n-th order, their unions have contact of the* $(n+1)$*-st order.*

Applying theorems 11 and 12 to a complex curve and its tangents, we obtain the following interesting correspondence:

THEOREM 13: *A complex curve and its tangents have the following interpretation in the plane:*

Complex curve.	Union.
∞^1 tangents.	∞^1 osculating circles.
P tangent.	Cusp.*
L tangent.	Inflectional tangent.*
Stationary tangent.†	Hyperosculating circle.‡

Knowing the distinct types of curves under G_{10},§ we obtain the following classification of line-element loci of ∞^1 elements with respect to Γ_{10}.§

* Note that the simple point and line singularities on a union arise not from singularities on the corresponding complex curve in space, but from the relation of the latter to the P and L lines of the complex. For higher singularities on the union, however, the complex curve itself must possess singularities. For example, a node on the complex curve gives rise to a tac-node on the union.

† The condition for a stationary point (cusp) on a complex curve is given by $g'f''-g''f'=0$. See Picard, *Annales de l'École Normale Supérieure*, 2. *série*, 6.

‡ Note that a cusp on a complex curve does not in general give rise to a singularity, but an extraordinary point (vertex) on the union, unless the cuspidal tangent is a P line or an L line, in which case the union possesses higher point or line singularities.

§ See Introduction.

Space.	*Plane.*
I. The general skew curve.	I. *Skew** line-element locus.
II. The general plane curve. (See Th. 10.)	II. *Plane** line-element locus; that is, a general line-element locus of ∞^1 elements selected from a pencil of tangent circular unions.
III. The general line. (See Th. 2.)	III. *Linear** line-element locus; that is, one whose point locus and line locus are concentric circles.
IV. The general complex curve.† (See Th. 1.)	IV. The general union.
V. The complex line.‡ (See Ths. 4, 5, 6.)	V. Circular, point and line unions.§

§ 2. *Line-Element Loci of ∞^2 Elements.*

Consider a general surface in space defined by

$$F(X, Y, Z) = 0.$$

There is one complex line tangent to this surface at each point; viz., the intersection of the tangent plane and the complex plane|| at that point. There are then ∞^1 complex curves on a surface. The line of intersection becomes indeterminate in two cases:

(a) when the tangent plane is indeterminate;¶

(b) when the tangent plane and the complex plane coincide.

The first case is that of surface singularities, which we shall not consider in detail at this point, since it bears no especial relation to the geometry within the complex.**

* The various types are named after the corresponding curves in space.

† No complex curve except the straight line can be plane.

‡ P and L lines are not in themselves distinct under G_{10}.

§ These three types are not individually distinct.

|| That is, the plane determined by the null-system. See Introduction, (7).

¶ The complex plane is never indeterminate.

** It is to be remarked that if a surface possesses an n-tuple curve, then n complex curves on the surface pass through each point on this curve.

The second case is the more important in the work that follows. A point on a surface where the tangent plane and complex plane coincide will be referred to as a *singular complex point.* It must be clearly understood that such points are not surface singularities, but are singular only with respect to the *position* of the surface in the complex. If such points are finite in number or infinite and at the same time discrete, we shall call them *complex nodes.* If the locus of singular complex points on a surface is a continuous curve, we shall call the latter a *singular complex curve* of the surface, for it is indeed complex.

It is evident that an infinite number of complex curves on a surface intersect at its complex nodes, and that through each point on a singular complex curve there passes one other complex curve of the surface.

We wish to consider now the loci of such points on a general surface, where the complex tangent lines are P lines and L lines respectively. We shall call these loci the *P and the L curves* respectively of the surface.

Let the surface be defined by

$$F(X, Y, Z) = 0.$$

Let $P_0(X_0, Y_0, Z_0)$ be a point on the surface.

$$F(X_0, Y_0, Z_0) = 0.$$

The tangent plane at this point is defined by

$$F_{X_0} \cdot X + F_{Y_0} \cdot Y + F_{Z_0} \cdot Z - (X_0 F_{X_0} + Y_0 F_{Y_0} + Z_0 F_{Z_0}) = 0,$$

and the complex plane by

$$Y_0 X - X_0 Y - Z + Z_0 = 0.$$

The pencil of planes through their line of intersection is defined by

$$\begin{aligned}(F_{X_0} + K Y_0) X + (F_{Y_0} - K X_0) Y + (F_{Z_0} - K) Z \\ + (K Z_0 - X_0 F_{X_0} - Y_0 F_{Y_0} - Z_0 F_{Z_0}) = 0.\end{aligned} \tag{15}$$

This line of intersection is a P line if, for some value of K, the equation (15) reduces to

$$MZ + N = 0.$$

Hence,

$$F_{X_0} + K Y_0 = 0,$$
$$F_{Y_0} - K X_0 = 0.$$

Eliminating K from these equations and omitting subscripts, we obtain

$$X F_X + Y F_Y = 0. \tag{16}$$

We shall call the locus of this equation the *P surface of F.* Its curve of intersection with F, which is not at the same time a singular curve* of F, is evidently the P curve on the surface.

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* Note that the P surface passes through all the surface singularities of the original surface which are defined by $F_X = F_Y = F_Z = 0$. The complex tangents at such points are not in general P lines. Note also that equation (16) may reduce to an absurdity, in which case the P curve does not exist, or it may give an identity, in which case we have a ruled surface F all of whose generators are P lines. These remarks apply equally well to the L surface.

Similarly, we derive an equation

$$F_X + YF_Z = 0, \tag{17}$$

whose locus we shall call the *L surface of F*, and whose curve of intersection with F, which is not at the same time a singular curve, is the L curve of the surface.

Note that if both relations (16) and (17) are satisfied by the coördinates of a point on F, then the tangent plane and the complex plane at that point coincide (see (14)). Hence, the singular complex points on a surface are defined by the intersection of the three surfaces

$$\left.\begin{aligned} F(X, Y, Z) &= 0, \\ XF_X + YF_Y &= 0, \\ F_X + YF_Z &= 0. \end{aligned}\right\} \tag{18}$$

THEOREM 14: *The P and L curves of a surface pass through its complex nodes.*

If the three surfaces (18) have in common a curve which is not a surface singularity of F, that curve is the singular complex curve of F.

THEOREM 15: *A surface in general has no singular complex curve.*

The interpretation is the following:

Space.	*Plane.*
General surface.	General diff. equation of 1st order.
$F(X, Y, Z) = 0$.	$f(x, y, p) = 0$.
∞^1 complex curves of F.	∞^1 integral curves of f.
∞^2 complex tangents to F.	∞^2 osculating circles of integral curves.
P surface: $XF_X + YF_Y = 0$.	Diff. equation: $f_p = 0$.*
L surface: $F_X + XF_Z = 0$.	Diff. equation: $f_x + pf_y = 0$.*
P curve.	Point locus is locus of cusps on integral curves. †
L curve.	Line locus is envelope of inflectional tangents. ‡
Complex node.	Tac-point.
Singular complex curve.	Singular solution. §

* For analysis see foot-note in § 3.

btained by eliminating p from $f=0$ and $f_p=0$, the well-known p-discriminant locus. The line locus of the P curve is the envelope of cuspidal tangents.

‡ The point locus of the L curve is the locus of inflections.

§ The singular solution satisfies the three equations

$$f = 0, \qquad f_p = 0, \qquad f_x + pf_y = 0.$$

Note, however, that the P and L surfaces always intersect the original surface in its singular curves. A singular curve on a surface evidently gives rise to a tac-locus in the plane; the line elements along this tac-locus also satisfy the three equations above. Usually we think of the line elements of the singular solution alone as satisfying these relations.

Note that if the singular complex curve on a surface is a P line, the singular solution becomes a ray-point.

Since the geometry within the complex is projective, we shall consider first the complex characteristics of ruled surfaces and the corresponding line-element loci of ∞^2 elements in the plane. We shall return to the general differential equation later.

Since the tangent plane at any point on a ruled surface contains the generator through that point, it follows that the singular complex points on a ruled surface can lie only on those generators which are complex lines.* Let l be a complex generator; the complex planes as well as the tangent planes pass through l for all points on l. Hence, by the theory of involution, for two points at most on l, the two planes coincide. Having shown that the number of singular complex points on any complex generator is finite, we can state

THEOREM 16: *A general ruled surface can not have a singular complex curve.*

THEOREM 17: *There are two complex nodes on every complex generator of a general scroll.*

THEOREM 18: *There is one complex node on every complex generator of a general developable.*

Consider developable surfaces: First a torse; in passing over the cuspidal edge of a torse the complex curves on the surface must acquire a cusp.†

THEOREM 19: *The cuspidal edge on a general torse is the locus of stationary points on its complex curves.*

We shall now center our attention on ruled surfaces all of whose generators are complex lines; they are evidently complex scrolls, complex torses and planes (complex cones).

By Th. 17, we obtain

THEOREM 20: *There are two singular complex curves on every complex scroll.*‡

We shall refer to a complex scroll all of whose generators are P lines as a *P scroll,* § and to one all of whose generators are L lines as an *L scroll.* ||

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* There are always some complex generators on a general ruled surface. Let the surface be defined by the intersection of the two planes

$$p_6 x = p_4 z + p_2, \qquad p_5 x = p_4 y - p_3,$$

where the p's are functions of a single parameter t. The equation in t, $p_3 = p_6$, determines the complex generators.

† Unless the tangent to the curve at that point is tangent also to the edge, which is extraordinary.

‡ This theorem is due originally to Picard, *loc. cit.*

§ P scrolls are defined by $X = Yf(Z)$. See (13), § 1.

|| L scrolls are defined by $Z - YX = f(Y)$. See (14), § 1.

One of the singular complex curves on an L scroll is the line at infinity in the XZ plane. We wish to consider complex scrolls both of whose singular complex curves are complex lines. Select two non-intersecting complex lines in space. The locus of complex lines meeting both of these lines is by the theory of one-to-one correspondence a quadric surface. Conversely, consider a complex quadric. The tangent planes at the two complex nodes on a complex generator cut the quadric each in a second generator, which must also be complex; hence

THEOREM 21: *The two singular complex curves on every complex quadric are two generators of the other system.*

THEOREM 22: *There is one singular complex curve on a complex torse — its cuspidal edge.**

THEOREM 23: *The complex point in a plane is a complex node.*

It is evident that a complex scroll in space must give rise to a differential equation in the plane whose integral curves are circles: an L scroll, in particular, to one whose integral curves are straight lines, that is, a Clairaut type; while a P scroll goes over into ∞^1 point unions. Hence, we may state

THEOREM 24: *All space curves lying on the same P scroll have the same point locus.*

THEOREM 25: *All space curves lying on the same L scroll have the same line locus.*

We associate with a general surface, a P scroll and an L scroll which touch the surface along its P curve and L curve respectively. Hence, they both touch the surface at the points where these curves intersect; viz., at the complex nodes. It follows then that these points are singular complex points on the scrolls themselves, and hence

THEOREM 26: *The singular complex curves on the P and L scrolls of a surface pass through the complex nodes on the surface.*

Interpreted in the plane by means of Theorems 9, 24 and 25, this becomes

THEOREM 27: *The locus of cusps on the integral curves of a differential equation touches the envelope of their inflectional tangents at the tac-points.*

We wish now to classify differential equations (line-element loci of ∞^2 elements) with respect to Γ_{10}. We shall first enumerate the distinct types of surfaces in space under G_{10}. In classifying surfaces, we must notice that while

* Note here the extraordinary case of the complex curves on a surface being tangent to its singular complex curve.

all complex lines in space form an invariant system, the P and the L lines are not in themselves distinct types, and properties of a surface depending on these lines are not invariant. On the other hand, the complex singularities — the complex node and the singular complex curve — are invariant properties. Ruled surfaces all of whose generators are complex lines, since they always possess singular complex curves, can be classified directly from this point of view.

Complex Ruled Surface.	*Diff. Eq. whose Int. Curves are Circles.*
VI. General complex scroll with two general singular complex curves (IV).	VI. General one-parameter family of circles with a double envelope.
VII. Complex scroll one of whose singular complex curves is a complex line (V).	VII. One-parameter family of circles touching a curve and a fixed circle.*
VIII. Complex quadric. (See Th. 21.)	VIII. All the circles touching two fixed circles.*
IX. Complex torse. (See Th. 22.)	IX. All the circles osculating a given curve.
X. Plane. (See Th. 10.)	X. Pencil of tangent circles.†

We know that the remaining distinct types of surfaces under G_{10} are the general surface, the general scroll, the general torse and the general cone. By Theorem 16, none of the last three can have a singular complex curve, while the general surface may or may not possess one. But the distinguishing characteristics of the corresponding differential equations in the plane are obtained from a consideration of the arrangement of the ∞^2 complex lines in space which are tangent to the surface. Since these correspond to the osculating circles of the integral curves in the plane, we see that the basis of classification is the distinct arrangements under Γ_{10} of the ∞^2 circles which osculate the integral curves, and hence the significance of the title of the paper is realized.

* The two fixed circles may also be points or lines.

† Note that all the line elements through the tac-point satisfy the differential equation (see Th. 10).

Now if the surface is a developable, then the tangent complex lines lie in its ∞^1 tangent planes, which clears up the arrangement for the torse and the cone — the latter being distinguished from the former in that its tangent planes have a point in common. For the general surface the arrangement is general. The case of a general scroll needs more consideration. We fix our attention on one generator; the complex lines tangent to the surface along this generator meet also a second line (the reciprocal polar), and hence are the generators of a complex quadric.

XI.

General surface: Tangent lines have no particular arrangement.

XII.

General scroll: The ∞^2 complex tangent lines can be grouped as the generators of ∞^1 complex quadrics which touch the scroll each along a generator.*

(See Types III and VIII.)

XIII.

General torse: The ∞^2 complex tangent lines lie in ∞^1 planes.

(See Type X.)

XIV.

General cone: The ∞^2 complex tangent lines lie in ∞^1 planes having a point in common.

XI.

General differential equation: The osculating circles have no particular arrangement.

XII.

Scroll† *differential equation:* The ∞^2 osculating circles can be grouped into ∞^1 families. The circles of each family touch two other circles and osculate the integral curves along a circle. The tangents to the integral curves along the latter circle envelope a concentric circle.

XIII.

Torse differential equation: The ∞^2 osculating circles can be grouped into ∞^1 pencils of tangent circles.

XIV.

Cone differential equation: The arrangement is similar to that in XIII, with the specialization that there is one pencil having one circle in common with each of the other pencils.

The properties here given of these four types of differential equations sufficiently characterize them; but they have additional properties by far more interesting, reserved for the following section.

* This last clause is very important, since the envelope of ∞^1 ruled surfaces is not necessarily a ruled surface.

† We shall name the differential equation after the corresponding surface.

§ 3. *Reciprocation.*

The linear line complex Γ is invariant also under the correlation which it defines when regarded as a null-system, for then each line of Γ is itself invariant. This fact gives rise to a theory of reciprocal line-element loci in the plane.

Let us recall the notion of reciprocation in space. Consider a general surface S of ∞^2 points and of ∞^2 tangent planes. Then under the null-system (7) we associate ∞^2 complex planes with the ∞^2 points on S; they envelope a surface $\bar{S}$ which is also the locus of the ∞^2 complex points of the tangent planes of S. We refer to S and $\bar{S}$ as reciprocal surfaces under Γ.

The points on S and $\bar{S}$ are in a one-to-one correspondence such that if P and $\bar{P}$ are corresponding points, the tangent plane to S at P is the complex plane of $\bar{P}$, and the complex plane of P is the tangent plane to $\bar{S}$ at $\bar{P}$. The line joining P and $\bar{P}$ is evidently a complex line tangent to both surfaces as well as to the integral curve on each surface passing through its points of contact respectively. The reciprocation may be looked upon as a transformation which takes one point on a complex line over into a second point on the same line.

In the plane, there are two differential equations D and $\bar{D}$ corresponding to S and $\bar{S}$ respectively. Corresponding to the line $P\bar{P}$ is a circle which osculates an integral curve of D and of $\bar{D}$ on the line elements e and $\bar{e}$ respectively. That is, in the plane we have a transformation which takes each line element over into a second line element on the same circle.

Let the equation of the surface S be

$$F(X, Y, Z) = 0. \tag{19}$$

The equations of transformation in space are

$$\left.\begin{aligned} \bar{X} &= \frac{F_Y}{F_Z}, \\ \bar{Y} &= -\frac{F_X}{F_Z}, \\ \bar{Z} &= Z + \frac{X F_X}{F_Z} + \frac{Y F_Y}{F_Z}, \end{aligned}\right\} \tag{20}$$

and the equation of the reciprocal surface $\bar{S}$ is found by eliminating X, Y and Z from equations (19) and (20).

Let D have the form

$$f(x, y, p) = 0. \tag{21}$$

The equations of transformation in the plane are

$$\left.\begin{aligned}\bar{x} &= x + \frac{2(1+p^2)f_p f_y}{f_x^2+f_y^2},\\ \bar{y} &= y - \frac{2(1+p^2)f_p f_x}{f_x^2+f_y^2},\\ \bar{p} &= \frac{p(f_y^2-f_x^2)-2f_xf_y}{(f_y^2-f_x^2)-2pf_xf_y},\end{aligned}\right\} \quad (22)^*$$

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* From the Lie transformation

$$\left.\begin{aligned}x &= i\left(Z+\frac{X}{Y}\right),\\ y &= Z-\frac{X}{Y},\\ p &= i\,\frac{(1-Y^2)}{1+Y^2},\end{aligned}\right\} \quad (23)$$

we obtain the following relations:

$$\left.\begin{aligned}Z &= \frac{x+iy}{2i}\\ \frac{X}{Y} &= \frac{x-iy}{2i},\\ Y^2 &= \frac{i-p}{i+p},\\ \frac{4Y^2}{(1+Y^2)^2} &= 1+p^2.\end{aligned}\right\} \quad (24)$$

Now, from equations (20) we obtain

$$\left.\begin{aligned}\bar{Z} &= Z+\frac{XF_X}{F_Z}+\frac{YF_Y}{F_Z},\\ \frac{\bar{X}}{\bar{Y}} &= -\frac{F_Y}{F_X},\\ \bar{Y}^2 &= \frac{F_X}{F_Y^2}.\end{aligned}\right\} \quad (25)$$

Assuming $F(X, Y, Z) = f(x, y, p)$, we obtain, by reference to (23),

$$\left.\begin{aligned}F_X &= f_x\frac{\partial x}{\partial X}+f_y\frac{\partial y}{\partial X}+f_p\frac{\partial p}{\partial X} = \frac{1}{Y}(if_x-f_y),\\ F_Y &= \frac{X}{Y^2}(f_x-if_y)-if_p\left[\frac{4Y}{(1+Y^2)^2}\right],\\ F_Z &= if_x+f_y.\end{aligned}\right\} \quad (26)$$

Equations (25) then, by reference to (24) and (26), become

$$\left.\begin{aligned}\frac{\bar{x}+i\bar{y}}{2i} &= \frac{x+iy}{2i}-\frac{i(1+p^2)f_p}{if_x+f_y},\\ \frac{\bar{x}-i\bar{y}}{2i} &= \frac{x-iy}{2i}+\frac{i(1+p^2)f_p}{if_x-f_y},\\ \frac{i-\bar{p}}{i+\bar{p}} &= \frac{(if_x-f_y)^2(i+p)}{(if_x+f_y)^2(i-p)}.\end{aligned}\right\} \quad (27)$$

Adding and subtracting the first two equations and solving the last for $\bar{p}$ we obtain equations (22).

and the *reciprocal differential equation* is found by eliminating x, y and p from equations (21) and (22).

Let us collect here some of the properties of a general surface and its reciprocal.

(a) The ∞^2 complex tangents to one are also the ∞^2 complex tangents to the other; or in other words, the complex curves on both surfaces have the same ∞^2 tangents. (See Th. 13)

(b) The P lines tangent to one are also tangent to the other; that is, the P curves of both surfaces lie on the same P scroll. (See Th. 24.)

(c) The L lines tangent to one are also tangent to the other; that is, the L curves of both surfaces lie on the same L scroll. (See Th. 25.)

(d) The complex nodes on one surface are also complex nodes on the other, and the two surfaces touch at these points. (See Th. 27.)

(e) If one surface possesses a singular complex curve, the other possesses the same singular complex curve; the two surfaces touch along this curve and the complex curves on one surface are tangent one by one to the complex curves on the other along this curve. (See Th. 11.)

The interpretation in the plane is as follows:

THEOREM A: *Associated with a general ordinary differential equation of the first order is a reciprocal differential equation such that:*

(a)* *The ∞^2 osculating circles to the integral curves of one osculate also the integral curves of the other.*

(b) *Both sets of integral curves have the same locus of cusps.*

(c) *The inflectional tangents to both sets of integral curves have the same envelope.*

(d) *This envelope touches the locus of cusps, and the points of contact are common tac-points on both sets of integral curves.*

(e) *If one differential equation has a singular solution, the other has the same singular solution, and the integral curves of one osculate those of the other along this common envelope.*

Consider a general scroll in space. Its reciprocal surface is a second scroll, whose generators are the conjugate polars of the generators of the first, such that:

(f) The complex lines tangent to one scroll along a generator are tangent

* Each property in the plane follows from the property in space with the corresponding letter given above, where the theorems, which enable us to make the interpretation, are also referred to.

to the reciprocal scroll also along a generator which is the conjugate polar of the first. (See Ths. 2, 3, 21; also Type VIII.)

(g) The complex generators on one scroll are also complex generators on the other. (See Ths. 4, 17.)

THEOREM B: *Associated with every scroll differential equation is a second reciprocal scroll differential equation, and in addition to having the properties* (a), (b), (c) *and* (d)* *stated in Theorem* A, *they are so related that:*

(f) *Their* ∞^2 *common osculating circles can be grouped into* ∞^1 *families. The circles of each family touch two other circles and osculate the integral curve of both differential equations along the same circle. The tangents to both sets of integral curves along this circle envelope a common concentric circle.*

(g) *The two differential equations have in common a number of integral curves which are circles, and an infinite number of integral curves of both differential equations touch these circles at two points.*

Consider a developable surface in space. Since it has but a single infinity of tangent planes, its reciprocal surface must degenerate into a locus of ∞^1 points — hence a curve. We shall say, then, that such surfaces have degenerate reciprocals. It is evident that the reciprocals of the corresponding differential equations in the plane are then line-element loci of ∞^1 elements. We shall say that such differential equations have degenerate reciprocals.† This fact does not, however, detract from the interest in such equations, but rather adds, for the line-element locus of ∞^1 elements has very important relations to the integral curves.

Consider in particular a general torse:

(h) Its reciprocal is a *skew* curve which is met by all the ∞^2 complex lines tangent to the surface. (See Type I.) Hence

(i) The P curve on a torse and the reciprocal skew curve of the latter lie upon the same P scroll. (See Th. 24.)

(j) The L curve on a torse and the reciprocal skew curve of the latter lie upon the same L scroll. (See Th. 25.)

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* Property (e) is ruled out by Theorem 16.

† The analytic condition that the reciprocal of a differential equation $f(x, y, p) = 0$ be degenerate is expressed by the vanishing of the functional determinant in the equations of transformation (22).

(l) Since the tangent plane touches a torse along a generator, it follows that the complex lines lying in that plane are tangent to the complex curves of the torse along a generator. (See Ths. **10**, 2 and **11**.)

(k) Since each generator meets the cuspidal edge of the torse, and since the complex curve at that point has a cusp, it follows that one complex line in each plane is a stationary tangent to a complex curve. (See Th. **13**.)

(m) The generators of the torse (*i. e.*, the tangents to its cuspidal edge) are the conjugate polar lines of the tangents to its reciprocal skew curve. (See Ths. 2, 3, 8.)

The interpretation in the plane follows:

THEOREM C: *A torse differential equation has the following properties:*

(h) *The ∞^2 osculating circles of its integral curves can be grouped into ∞^1 pencils of tangent circles, the common line element in each pencil giving rise to a skew line-element locus.*

(i) *The point locus of this line-element locus is the locus of cusps on the integral curves.*

(j) *The line locus is the envelope of inflectional tangents to the integral curves.*

(k) *There is one circle of each pencil which hyperosculates an integral curve, thus giving rise to a locus of hyperosculation.**

(l) *The circles of each pencil osculate the integral curves along a circle, and the tangents to the integral curves along the latter envelope a concentric circle.*

(m) *The first set of circles thus determined are enveloped by the cuspidal locus and the locus of hyperosculation, while the concentric circles touch both the envelope of inflectional tangents and the envelope of tangents drawn along the locus of hyperosculation.*

We associate with every general torse in space a second torse defined by the envelope of the ∞^1 complex planes of the points on the cuspidal edge of the first. We shall refer to these as *related* torses. They have the following properties:

(n) The cuspidal edge of one is the reciprocal skew curve of the other.

(o) The complex lines tangent to one along its cuspidal edge are tangent also to the other along its cuspidal edge; that is, the ∞^1 stationary tangents to

* The second line-element locus of ∞^1 elements thus determined is that corresponding to the cuspidal edge in space.

the complex curves on one torse are also stationary tangents to the complex curves on the other. (See Th. 13.)

(p) Consider one of these common stationary tangents. Let it meet the cuspidal edge of the first torse at P and the cuspidal edge of the related torse at P'. Then the complex lines through P touch the related torse along that generator which is tangent to its cuspidal edge at P', and the complex lines through P' touch the other torse along that generator which is tangent to its cuspidal edge at P. These two generators are conjugate polar lines. (See Ths. 2, 3 and 8.)

The interpretation follows:

THEOREM D: *Associated with every torse differential equation is a related* torse differential equation such that:*

(n) *The cuspidal locus of the integral curves of one is the locus of hyperosculation on the integral curves of the other, and the envelope of inflectional tangents to the integral curves of one is the envelope of the tangents drawn to the integral curves along the locus of hyperosculation of the other.*

(o) *The* ∞^1 *circles which hyperosculate the integral curves of one hyperosculate also the integral curves of the other.*

(p) *Let* C *be one of these common hyperosculating circles and let* P *and* P' *be the two points of hyperosculation. Consider the two pencils of circles tangent to* C *at* P *and* P' *respectively. The circles of one pencil osculate the integral curves of one differential equation, and the circles of the second pencil osculate the integral curves of the other along the same circle* C_1. *The tangents drawn to both sets of integral curves along* C_1 *envelope a common concentric circle* C_2. *The* ∞^1 *circles* C_1 *are enveloped by both cuspidal loci, while the concentric circles* C_2 *touch both envelopes of inflectional tangents.*

It is interesting to note that the entire configuration discussed in the last theorem is built upon a single skew† line-element locus of ∞^1 elements, just as the configuration of related developables in space is built upon a single skew curve — the cuspidal edge of either one of them. Since one cuspidal edge

* Not reciprocal.

† It is essential that the line-element locus be skew and not plane. (See Theorem E.)

uniquely determines the other, so in the plane we associate a second skew line-element locus with the first. The two sets of pencils of tangent circles determined by these line elements are respectively the osculating circles to the integral curves of the related torse differential equations discussed in Theorem D.

Consider now a cone:

(q) Its reciprocal is a *plane* curve which is met by all the ∞^2 complex lines tangent to the surface. These include also those complex lines lying in the complex plane of the vertex. (See Type II.)

See (i) and (j) under torse.

(r) The complex lines lying in a tangent plane are tangent to the complex curves on the cone along an element. (See Ths. 10, 2 and 11.)

(s) The elements of the cone all pass through its vertex (see Th. 7) and are the reciprocal polars of the tangents to its reciprocal plane curve. (See Ths. 2, 3 and 8.)

The interpretation is as follows:

THEOREM E: *A cone differential equation has the following properties:*

(q) *The ∞^2 osculating circles of its integral curves can be grouped into ∞^1 pencils of tangent circles. One circle from each pencil belongs to a special pencil. The common line elements of these pencils thus give rise to a plane line-element locus and a particular isolated line element.**

See (i) and (j) of Theorem C.

(r) *The circles in each general pencil osculate the integral curves along a circle, and the tangents to the integral curves along the latter envelope a concentric circle.*

(s) *The first set of circles thus determined all touch the cuspidal locus and pass through the point of the particular line element, while the concentric circles all touch the envelope of inflectional tangents and the line of the particular line element.*

Up to this point we have considered the reciprocals of general surfaces, scrolls, torses and cones together with the corresponding interpretation in the plane. This has also involved the reciprocation of non-complex curves in space. The interpretation of complex ruled surfaces and their reciprocals gives rise to

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* Evidently the line element corresponding to the vertex of the cone.

nothing of particular interest in the plane from the standpoint of the ordinary differential equation of the first order, for the osculating circles in this case reduce to the integral curves themselves. However, the reciprocation of all the distinct types of surfaces and curves will be found to be equally important in the discussion of complex congruences and their interpretation which will appear in a later paper.

REMARK. It is evident that the geometry within a linear line complex can be equally well interpreted by other transformations aside from the one used in this paper. (See "Geometrie der Berührungstransformationen," p. 238.) But this one is perhaps the most important, since it contributes to the theory of the osculating circle.

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